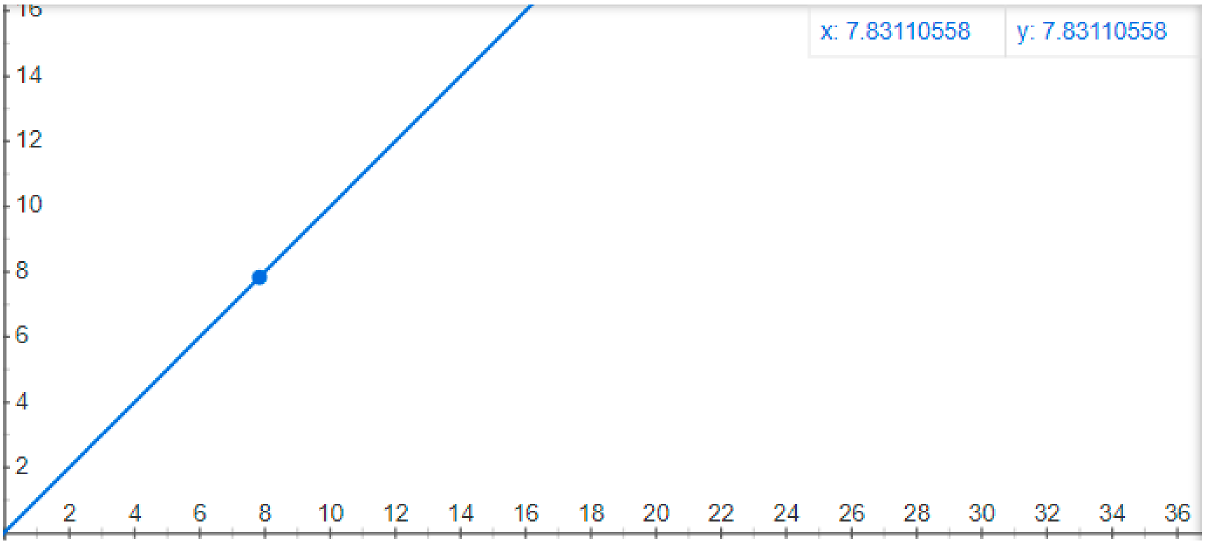
Derivatives and Integrals

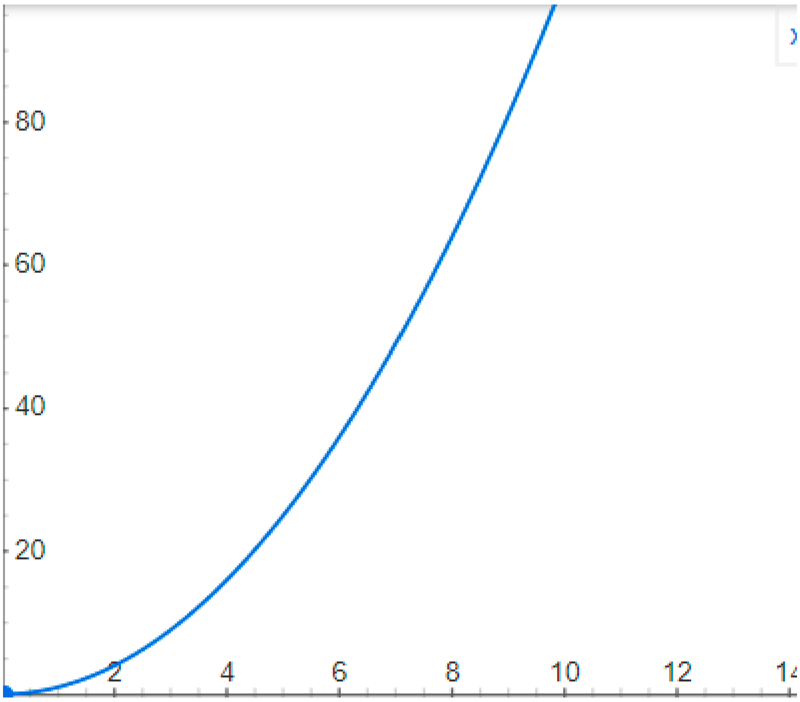
Let's say we have a line function



f(x)=x

Then we can say that the slope is $\frac{rise}{run}$

But what if it was a curve? $f(x) = x^2$



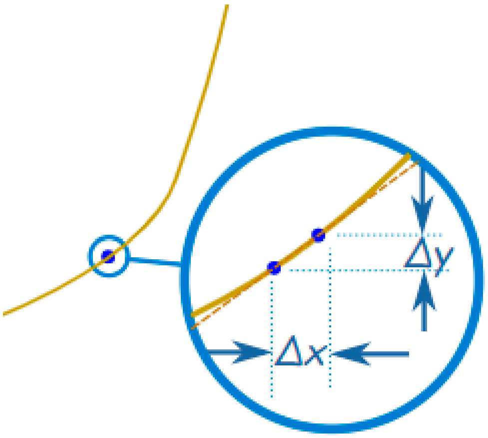
How do we get a slope? We can choose 2 points and use the rise over run formula, but that won't be accurate. That's why we get a slope of a point.

But let’s say we get the point: (2,4)

The change in Y in the point is 0 and change in X is also 0

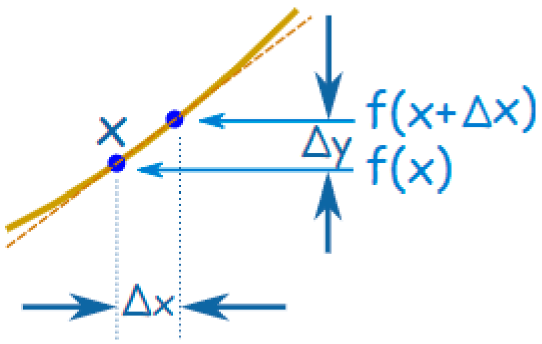
So for this we can get 2 points which are very very close to each other and get their slope and we can saw that as the distance becomes 0 between those two super close points we have out, well slope of a point.

So, we use a very small difference and then shrink it towards 0



Let’s find their slope:

The formula is $\frac{rise}{run}$ $or \frac{change\ in\ y}{change\ in\ x}$.



In the picture above we see that $x \to \Delta x$ and $y \to \Delta y$

Our rise would be $f(x + \Delta x) - f(x)$

And our run would be Δx

Therefore, our slope formula would be:$\frac{f(x+\Delta x)-f(x)}{\Delta x}$

Let’s try to solve this for $f(x) = x^2$

$$f(x + \Delta x) = (x + \Delta x)^2$$

$$x^2 + 2x\Delta x + (\Delta x)^2$$

Now let's put that in our formula:

$$\frac{x^2 + 2x\Delta x + (\Delta x)^2 - (x)^2}{\Delta x}$$

x^2 and $-(x^2)$ cancel out, then we divide by Δx , and that would give us:

$$2x + \Delta x$$

So as Δx heads towards 0 , the derivative of x^2 is $2x$.

We write dx instead of Δx heads towards 0.

And "the derivative of" is commonly is written as: $\frac{d}{dx}$

And therefore, $\frac{d}{dx}x^2 = 2x$

But what does $\frac{d}{dx}x^2 = 2x$ means? It means for the function $f(x) = x^2$ the "rate of change" in slope is $2x$

For $f(x) = x$^2 it's derivative: $\frac{d}{dx}x^2$ can also be written as: $f'(x)$

Derivatives Common Functions:

Common Functions	Function	Derivative
Constant	c	0
Line	x	1
Line with co efficient	ax	a
Square	x^n	nx^(n-1)
Exponetial	e^x	e^x
	a^x	ln(a)*a^x
Logarithms	ln(x)	1/x
Trigonometry(x -> rad)	sin(x)	cos(x)
	cos(x)	(-sin(x))
Inverse Trignometry	sin^-1(x)	1/sqrt(1-x^2)
	cos^-1(x)	−1/√(1−x^2)
	tan^-1(x)	1/(1+x^2)

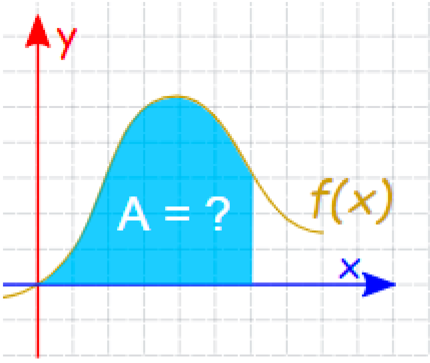
Rules of Derivatives

Rules	Funtions	Derivative
Multiplication by Const	cf	cf'
Power rule	x^n	nx^n-1
sum rule	f+g	f'+g'
Difference rule	f-g	f'-g'
Product rule	fg	fg'+f'g
division rule	f/g	(f'g-g'f)g^2
Reciprocal rule	1/f	(-f'/f^2)

The above-mentioned rules and common functions are a must to remember and very important for derivates and integrals.

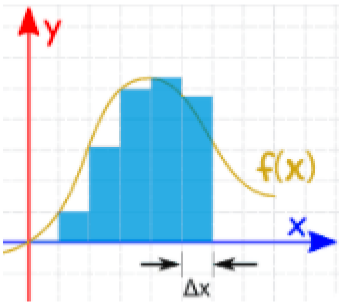
Integrals

Integration can be used to find areas, volumes, central points and many useful things. But it is easiest to start with finding the area between a function and the x-axis like this:

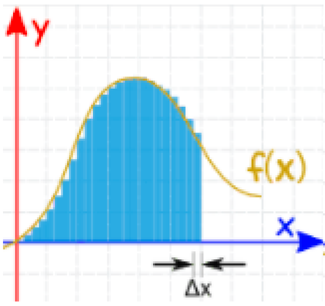


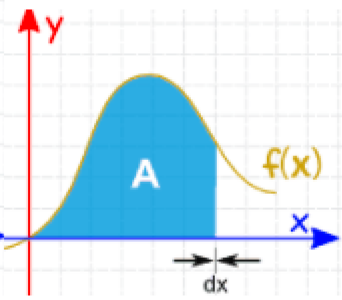
How to we calculate the area of this function?

We could, of course, slice up parts till width Δx, like this:



We can decrease the width of the slices less and less and after adding up the area of all slices we would approach the area of the function, but we would never get the exact value





We can say that as the width of each slice approaches 0 we have the area of our function.

But that would be A LOT of adding up, so this is where integrals come in.

Integrals are the opposite of derivatives.

So, as we calculate above: $\frac{d}{dx}x^2 = 2x$ the Integral of $2x$ is $x^2 + C$

The symbol for Integrals is a “S”, as in Sum, as in Summing up of slices.

$\int 2xdx = x^2 + C$; dx notates Slices along x

If you noticed above, we have a $+C$, but why?

Well C denotes the Constant of Integration.

It’s because “something’s” derivative can have a lot of Integrals.

$$\frac{d}{dx}x^2 + 4 = 2x$$

$$\frac{d}{dx}x^2 + 99 = 2x$$

And so on.

So, when we reverse the derivative to get the integral, there could have been any constant. And so, we write +C at the end.

Integration Common Functions

Common Functions	Function	Integral
Constant	$\int a\,dx$	$ax + C$
Variable	$\int x\,dx$	$x^2/2 + C$
Square	$\int x^2\,dx$	$x^3/3 + C$
Reciprocal	$\int (1/x)\,dx$	$\ln\lvert x\rvert + C$
Exponential	$\int e^x\,dx$	$e^x + C$
	$\int a^x\,dx$	$a^x/\ln(a) + C$
	$\int \ln(x)\,dx$	$x\ln(x) - x + C$
Trigonometry (x in radians)	$\int \cos(x)\,dx$	$\sin(x) + C$
	$\int \sin(x)\,dx$	$-\cos(x) + C$
	$\int \sec^2(x)\,dx$	$\tan(x) + C$

Rules of Integration

Multiplication by constant	$\int cf(x)\,dx$	$c\int f(x)\,dx$
Power Rule (n≠−1)	$\int x^n\,dx$	$\frac{x^{n+1}}{n+1} + C$
Sum Rule	$\int (f + g)\,dx$	$\int f\,dx + \int g\,dx$
Difference Rule	$\int (f - g)\,dx$	$\int f\,dx - \int g\,dx$

Limits

Let's say you have a functions: $f(x) = \frac{(x^2-1)}{(x-1)}$, how would you calculate $f(1)$

Let's try to solve it:

$$\frac{(1^2 - 1)}{1 - 1}$$

$$\frac{1-1}{1-1}$$

Which is just $\frac{0}{0}$ and that is impossible.

This is where limits come in: we can calculate the value of $f(x)$ as x approaches 1, not 1 just approaches 1.

So first we get: $f(0.5)$, then $f(0.9)$, $f(0.99)$, this way

Cannot get the exact value of $f(1)$ but we can find the value where x is very very close to 1.

x		$\frac{(x^2-1)}{(x-1)}$
0.5		1.50000
0.9		1.90000
0.99		1.99000
0.999		1.99900
0.9999		1.99990
0.99999		1.99999
...		...

Here we see the close x get to 1, it's value get's closer to 2.

We just solved a limit.

This is how you would write the limit:

$$\lim_{n\to 1} \frac{n^2-1}{n-1} = 2$$

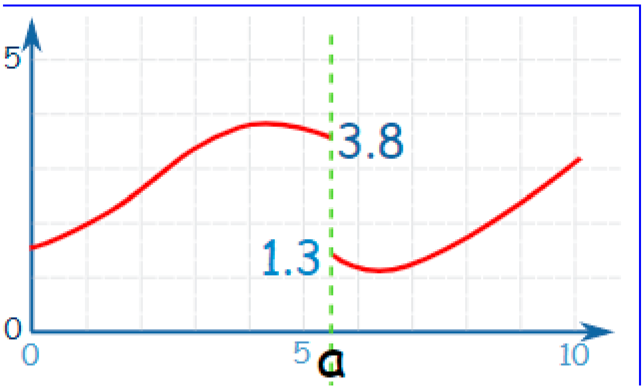
So as n approaches 1 the value of $\frac{n^2-1}{n-1}$ is 2.

But why only check from one side? Why not approach 1 from 2 and check it.

x	$\frac{(x^2 - 1)}{(x - 1)}$
1.5	2.50000
1.1	2.10000
1.01	2.01000
1.001	2.00100
1.0001	2.00010
1.00001	2.00001
...	...

The value is heading towards 2, and so we were correct.

But this was just a line function. What is it was a wave function and there was a huge break?



The value is 3.8 from the left and 1.3 from the right? Does this mean we have 2 answers? Or does it have no correct answer? Yes, and Yes, sort of. In cases like this we use + and – symbols.

$$\lim_{x \to a^-} f(x) = 3.8$$

And,

$$\lim_{x \to a^+} f(x) = 1.3$$

But,

$\lim_{x \to a} f(x)$ does not exist.

Infinity

Infinity is a special idea, a concept, not a real number. We think of infinity as travelling on and on but never reaching the destination, well that's wrong.

If something has no reason to end, it's infinite.

Infinity is thought to be growing on and on, but infinity is already fully formed.

Approaching Infinity

We dealt with limits approaching constants but what about infinity? How do we calculate with infinity?

How can we calculate: $\frac{1}{\infty}$?

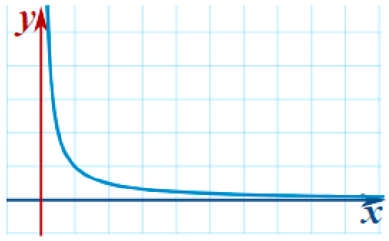
We can't! We just can't. Why? Because infinity is NOT A NUMBER, just an idea.

This is like dividing by 0, it's undefined.

But, there is a catch.

We can approach infinity.

x	$\frac{1}{x}$
1	1.00000
2	0.50000
4	0.25000
10	0.10000
100	0.01000
1,000	0.00100
10,000	0.00010



We can't say that what would happen to x when it reaches infinity. But we can say that $\frac{1}{x}$. We want to say $\frac{1}{\infty} = 0$ but it isn't, that's why we use limits.

$$\lim_{x\to\infty} \frac{1}{x} = 0$$

Evaluating Limits

We've learned a lot about limits but how are they calculated?

Firstly, we can't just try and write numbers closer and closer to the limit end. Like we did for calculating $\frac{x^2-1}{x-1}$

Another great way would be to use factoring

$$\lim_{x\to 1} \frac{(x^2-1)}{x-1}$$

$$(x^2-1) = (x+1)(x-1)$$

$$\lim_{x\to 1} \frac{(x-1)(x+1)}{x-1} = \lim_{x\to 1}(x+1)$$

Now we can just substitute x for 1 and see what happens.

$$\lim_{x\to 1}(x+1) = 1+1 = 2$$

Another way to evaluate limits is: Conjugate

Conjugate is when we change the sign of an expression:

Let’s do an example for a better understanding.

$$\lim_{x \to 4} \frac{2 - \sqrt{x}}{4 - x}$$

Here, if x was 4, it would give us $\frac{0}{0}$ which is undefined.

Let’s try solving it by conjugations.

Let’s multiply the top and bottom sides with $2 - \sqrt{x}$

$$\frac{2 - \sqrt{x}}{4 - x} * \frac{2 + \sqrt{x}}{2 + \sqrt{x}}$$

$$\frac{2^2 - (\sqrt{x})^2}{(4-x)(2+\sqrt{x})} => \frac{4-x}{(4-x)*(2+\sqrt{x})}$$

$$=> \frac{1}{2 + \sqrt{x}}$$

Now we can substitute x for 4.

$$\frac{1}{2 - \sqrt{4}} => \frac{1}{4}$$

Therefore, $\lim_{x \to 4} \frac{2-\sqrt{x}}{4-x} = \frac{1}{4}$

Lastly, a great method, L’Hopital’s rule:

This way of solving is a great for limits that seem indeterminate.

Let’s get straight to it with an example.

$$\lim_{x \to \infty} \frac{x^2}{e^x}$$

If we try, $x = \infty$ then we would get $\frac{\infty}{\infty}$ which is indeterminate.

Well, according to L’Hopital’s rule, if a limit evaluates to indeterminate then we replace the function with it’s derivative.

That is,

$$\lim_{x \to C} \frac{f(x)}{g(x)} => \lim_{x \to C} \frac{f'(x)}{g'(x)}$$

We can try this formula with $\frac{x^2}{e^x}$

$$\lim_{x\to\infty} \frac{(x^2 \frac{d}{dx})}{\frac{e^x d}{dx}}$$

Derivative of x^2 is $2x$.

And The derivative of e^x is e^x, fun fact Euler's number (e) is the only number which if powered to x, is it's own derivative. $\frac{d}{dx} e^x = e^x$

Back to solving the limit:

$$\lim_{x\to\infty} \frac{2x}{e^x}$$

This is still indeterminate, so we get their derivatives again.

$$\lim_{x-\infty} \frac{2}{e^x} => \frac{2}{\infty} = 0$$

So:

$$\lim_{x\to\infty} \frac{x^2}{e^x} = 0$$

We solved the above limit with L'Hopital's rule.