Question:

How many solutions does the equation?

$$\sin\left(\frac{\pi}{2}\cos(x)\right) = \cos\left(\frac{\pi}{2}\sin(x)\right)$$

have in the closed interval $[0, \pi]$?

Solution:

To solve this problem, we must first simplify the equation. Let's look at some different ways to write Sine and Cosine functions.

- Sine:

$$\sin(x) = \frac{opposite}{hypotenuse}$$

$$\sin(x) = -\left(\frac{d}{dx}\cos(x)\right)$$

$$\sin(x) = \cos\left(\frac{\pi}{2} - x\right)$$

- Cosine:

$$\cos(x) = \frac{adjacent}{hypotenuse}$$

$$\cos(x) = \frac{d}{dx}\sin(x)$$

$$\cos(x) = \sin\left(\frac{\pi}{2} - x\right)$$

Out of the above 3 equations per function,

$$\sin(x) = \cos\left(\frac{\pi}{2} - x\right)$$

And

$$\cos(x) = \sin\left(\frac{\pi}{2} - x\right)$$

Seem for suitable for simplifying the equation.

This is because dealing with length of a side of triangle will seem more problematic as we aren't working with Triangles. And so will using derivates.

In addition, using the chosen formulas where we have 2 different composite functions, having all function is cos or sin will make it must simpler and elegant.

Let's choose cos to be the outside function and see what it yields us.

Note: Let's ignore the interval for now as to make things less complicated.

$$\sin\left(\frac{\pi}{2}\cos(x)\right) = \cos\,(\frac{\pi}{2}\sin(x))$$

Let's change the right left side of the equation first.

$$\sin\left(\frac{\pi}{2}\cos(x)\right)$$

In this our input x is $\frac{\pi}{2}\cos\,(x)$

Let's replace this in $\cos\,(\frac{\pi}{2} - x)$ as this is equal to $\sin\,(x)$

$$\cos\,(\frac{\pi}{2} - (\frac{\pi}{2}\cos(x))$$

Therefore,

$$\sin\left(\frac{\pi}{2}\cos(x)\right) = \cos\,(\frac{\pi}{2} - (\frac{\pi}{2}\cos(x))$$

$$=> \sin\left(\frac{\pi}{2}\cos(x)\right) = \cos\,(\frac{\pi}{2} - \frac{\pi}{2}cos(x))$$

We now don't need to simplify the right side of the equation.

$$\cos\left(\frac{\pi}{2}\sin(x)\right)$$

When we put our new LHS and original RHS the cos isn't of any use anymore, thus simplifying the entire equation more. Let's see why.

$$\cos\left(\frac{\pi}{2} - \frac{\pi}{2} cos(x)\right) = \cos\,(\frac{\pi}{2}\sin(x))$$

As we need to find all x that satisfy the solution we can ignore the outer functions as they just give an output for y. Therefore, we need to find all x in an interval that satisfy this equation:

$$\frac{\pi}{2} - \frac{\pi}{2} cos(x) = \frac{\pi}{2}\sin\,(x)$$

We can still simply this equation,

Divide by $\frac{\pi}{2}$

$$\frac{\frac{\pi}{2} - \frac{\pi}{2} cos(x)}{\frac{\pi}{2}} = \frac{\frac{\pi}{2}\sin(x)}{\frac{\pi}{2}}$$

$$\frac{\left(\frac{\pi}{2} - \frac{\pi}{2} cos(x)\right) * 2}{\pi} = \frac{\left(\frac{\pi}{2}\sin(x)\right) * 2}{\pi}$$

This is what happens when dividing by pi and multiplying by 2

$\frac{\pi}{2} : \frac{\pi}{2} => \frac{\pi}{2} * \frac{2}{\pi} = 1$

That would make our equation

$$1 - \cos(x) = \sin\,(x)$$

We get 1 on the RHS,

$$-\cos(x) = \sin(x) - 1$$

We multiply both sides by -1

$$\cos(x) = -\sin(x) + 1$$

Get negative sin(x) to LHS,

$$\cos(x) + \sin(x) = 1$$

We need to find all x in an interval that satisfy this solution.

We start by drawing the plot of $\sin(x) + \cos(x)$ this is a relatively simply plot to draw, even by hand.

To simplify this, we will only draw the plot from $[0,2\pi]$, why not from $[0,\pi]$? Even thought this was the given interval we will end on 2π instead of π as this makes it easier to draw the addition of sine and cosine.

Drawing the plot of sin(x)

We know that the plot of sin(x) repeats periodically from $0\ to\ 2\pi$

Meaning the y values goes from 0, to its maximum value to 0 to its minimum values to 0. This is called one period as the y values repeats after this "period" or "cycle".

The min and max values are determined by the co-efficient before the function.

Meaning if we had $2\sin\,(x)$ the max and min will be 2 and -2 respectively. Here we have not co-efficient, which can also be written as the co-efficient of $\sin\,(x)$ is 1.

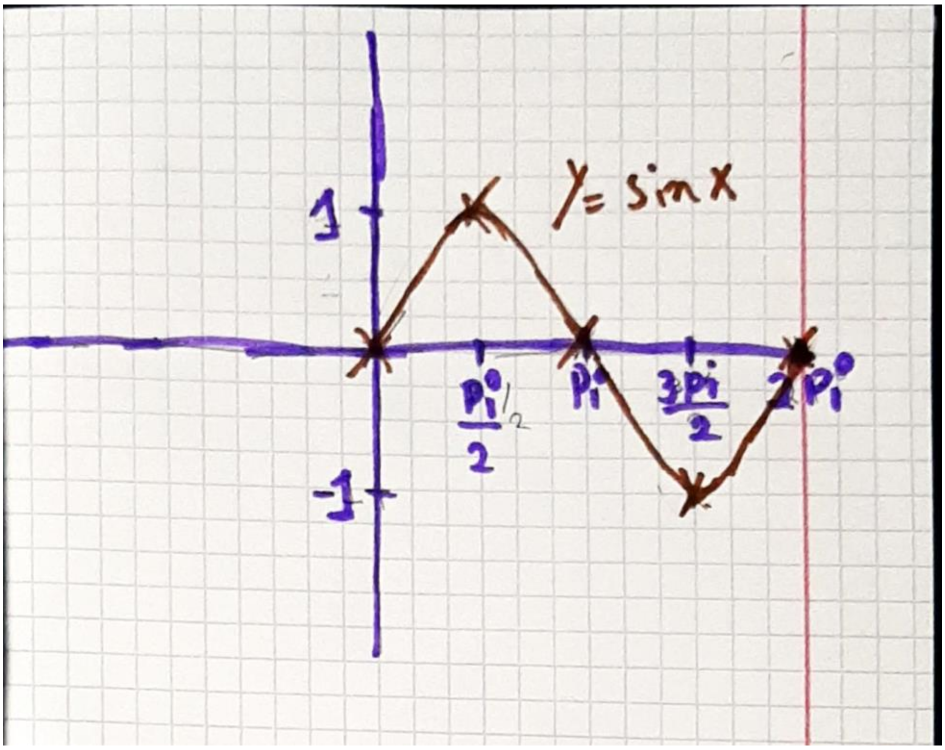
So, the max and min values of $\sin(x)$ will be 1 and -1, respectively.

Next, we know that $\sin\,(x)$ will have 3 zeros between $0\ and\ 2\pi$. The first zero will be at $x = 0$ and the last zero will be at $x = 2\pi$

As $\sin(x)$ is evenly spread out, the middle zero will be places at the midpoint of $0, 2\pi$, which is π

As sin(x) can reach max and min only once in the defined interval and the function is evenly spaced out, we know that it will reach it's peak or max between 0 and π and its valley or min between π and 2π

Now we can draw our function.



Let's do the same with cos(x).

It's like sin(x). It has a period of $[0,2\pi]$

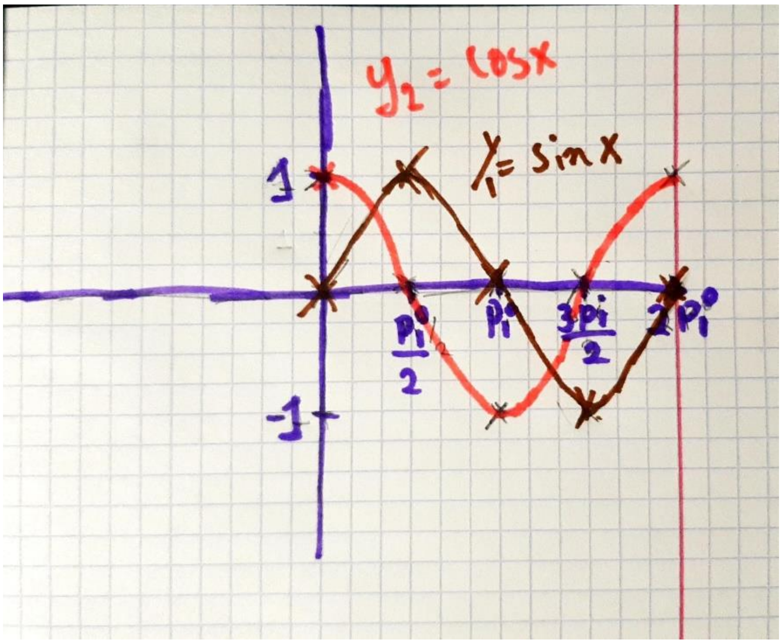
It also has it's peak at 1 and valley at -1.

But unlike sin(x), which has $y = 0 \; at \; x = 0$, cos(x) has$y = 1 \; at \; x = 0$. This also proves it will have only 2 zeros within it's period.

The first 0 will the midpoint of 0 and π and the next, midpoint of π and 2π

Which means the first 0 will pe $\frac{\pi}{2}$ and next at $\frac{3\pi}{2}$

Now we can plot the function cos(x). I'll be doing it on top of the sin(x) function as it'll we easier to add them.



Now, all we need to do it add the two functions and make the interval from 2π to π

We can start by adding the easy points.

First establishing a rule: if $\text{sinx} = \text{a}$ and $cosx = b$ then

$\sin(x) + \cos(x) = a + b$

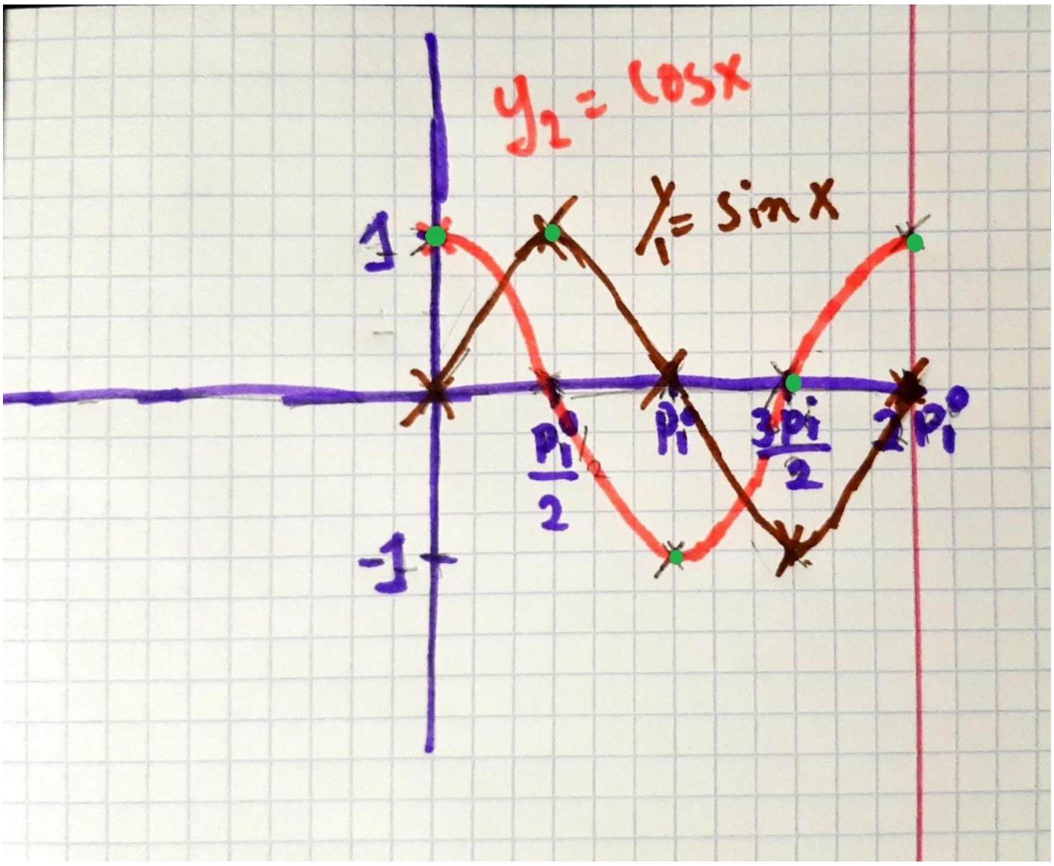
We can start by plotting the “easy” points.

For example, when $sinx = 0$ and $cosx = 1$, we know that

$\sin(x) + \cos(x) = 1$

Same when, $cosx = 0$ and $sinx = 1$, $\sin(x) + \cos(x) = 1$

Let’s mark all points when one of the functions yields 0.



Now we can change back to our original interval.

Which was $[0, \pi]$

Between the interval $[0, \pi]$ we see that there are only two instances when,

$$\cos(x) + \sin(x) = 1$$

Which are those green points on the graph. The above equation is only true when $x = 0 \vee x = \frac{\pi}{2}$

As we proved before that on an interval $[0, \pi]$

The values of x that satisfy:

$$\sin\left(\frac{\pi}{2}\cos(x)\right) = \cos\left(\frac{\pi}{2}\sin(x)\right)$$

And

$$\cos(x) + \sin(x) = 1$$

Are the same.

Answer:

Therefore, all values of x on an interval $[\mathbf{0}, \boldsymbol{\pi}]$ of the equation

$$\sin\left(\frac{\pi}{2}\cos(x)\right) = \cos\left(\frac{\pi}{2}\sin(x)\right)$$

Are $\boldsymbol{x} = \mathbf{0} \vee \boldsymbol{x} = \mathbf{pi/2}$